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Nonlinear contractions and Caputo tempered impulsive implicit fractional differential equations in b-metric spaces

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Abstract. This paper deals with some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit fractional differential equations in b-Metric spaces with initial nonlocal conditions and instantaneous impulses. The results are based on the $\omega - \delta$-Geraghty type contraction, the F-contraction and the fixed point theory. Furthermore, some illustrations are presented to demonstrate the plausibility of our results.

1. Introduction

Recently, fractional calculus has proven to be a valuable tool in addressing the intricacies inherent in various areas of research. It involves the extension of integer-order differentiation and integration to non-integer orders, and the study of its theory and applications has been substantial. For a comprehensive understanding, we recommend the reader to refer to the following resources: monographs such as [1,2,36,40] and papers such as [10,11,17,32,33]. There has been a surge of research on fractional calculus in recent years, with authors exploring a wide range of results for various forms of fractional differential equations and inclusions with different types of conditions. Further details can be found in papers such as [1, 3–5, 18, 23, 25, 26, 31, 34, 35] and their respective references.

Czerwik introduced the notion of b-metric [15, 16]. Following these early studies, the existence fixed point for various classes of operators in the context of b-metric spaces has been studied; see [7, 9, 14, 20, 30] for more details on the concept of b-metric and contractions.

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Wardowski [39] has asserted a novel inequality using auxiliary functions to ensure the existence and uniqueness of a particular mapping in the setting of standard metric space. This inequality is referred to as F-contraction. For more details on the $\omega-\delta$-Geraghty type contraction and the F-contraction, we refer the reader to the recent papers [9, 12, 24].

Tempered fractional calculus has emerged as an important class of fractional calculus operators in recent years. This class can generalize various forms of fractional calculus and possesses analytic kernels, making it an extension of fractional calculus that can describe the transition between normal and anomalous diffusion. The definitions of fractional integration with weak singular and exponential kernels were initially established by Buschman in [13], and further elaboration on this topic can be found in [8,19,27–29,37,38]. Although the Caputo tempered fractional derivative has not been extensively explored in the literature, it holds the potential to significantly contribute to this field. By studying this derivative, we aim to better understand its properties and potential applications in this unique mathematical notion, thus advancing fractional calculus.

In [22], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}_{t_k}^{\vartheta}x(t) = f\left(t, x_t, \mathcal{T}_k^{\vartheta}x(t)\right), & t \in J_k; k = 0, 1, \dots, \beta, \\ \Delta x|_{t=t_k} = \Upsilon_k(x_{t_k^-}), & k = 1, 2, \dots, \beta, \\ x(t) = \mu(t), & t \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = t_0 < t_1 < \cdots < t_\beta < t_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}_{t_k}^{\vartheta}x(t)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $f : J \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $J := [\varkappa, \bar{\varkappa}]$, $J_0 := [\varkappa, t_1]$, $J_k := (t_k, t_{k+1}]; k = 1, 2, \dots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_k : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

Motivated by the previously mentioned publications, in this paper, firstly we study the existence and uniqueness of solutions for the implicit problem with nonlinear fractional differential equation involving the Caputo tempered fractional derivative:

$$(1) \quad \begin{cases} \left({}_0^C\mathfrak{D}_t^{\alpha,\lambda}x\right)(t) = f\left(t, x(t), {}_0^C\mathfrak{D}_t^{\alpha,\lambda}x(t)\right), & t \in J_k; k = 0, 1, \dots, m, \\ \Delta x|_{t=t_k} = \hbar_k(x(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = x_0, \end{cases}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, ${}_0^C\mathfrak{D}_t^{\alpha,\lambda}$ is the Caputo tempered fractional derivative, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a given function to be specified later, $\hbar_k : \mathbb{R} \to \mathbb{R}$ are given continuous functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T < \infty$, $\Delta x\left|_{t=t_k}\right. = x(t_k^+) - x(t_k^-)$, $J := [0, T]$, $J_0 := [0, t_1]$, $J_k := (t_k, t_{k+1}]; k = 1, 2, \dots, m$, and $x_0 \in \mathbb{R}$.

Next, we discuss the existence of solutions for the following initial value problem of Caputo tempered implicit nonlocal fractional differential equations:

$$\begin{cases} \left({}_0^C\mathfrak{D}_t^{\alpha,\lambda}x\right)(t) = f\left(t, x(t), {}_0^C\mathfrak{D}_t^{\alpha,\lambda}x(t)\right), & t \in J_k; k = 0, 1, \ldots, m, \\ \Delta x|_{t=t_k} = \hbar_k(x(0t_k^-)), & k = 1, 2, \ldots, m, \\ x(0) + \overline{\delta}(x) = x_0, \end{cases} \tag{2}$$

where $\overline{\delta} : \mathcal{PC}(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and $\mathcal{PC}(J, \mathbb{R})$ is a space to be specified later (see Section 2).

As far as we know, the topic of implicit impulsive Caputo tempered fractional problems has not been extensively addressed in the existing literature. Given the scarcity of the published papers on tempered fractional calculus, there is a clear need for further research and advancement in this field. With this in mind, our aim is to contribute to the field by investigating problems that have not been studied yet with novel conditions. Additionally, our study intends to make use of new techniques, such as the F-contraction method, setting it apart from previous research such as [22].

The present study is structured as follows: In Section 2, we provide an overview of relevant background information and introduce key definitions, lemmas, and auxiliary results related to tempered fractional derivatives. Sections 3 and 4 focus on presenting existence and uniqueness results for the problems (1) and (2) respectively, utilizing $\omega - \delta$-Geraghty type contraction, F-contraction, and fixed point theory. Finally, in Section 5, we present a set of examples to demonstrate the validity of our findings.

2. Preliminaries

Firstly, we give the definitions and the notations that we will use throughout this paper. We denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the following norm

$$\|f\|_\infty = \sup_{t \in J}\{|f(t)|\}.$$

As usual, $AC(J, \mathbb{R})$ denotes the space of absolutely continuous functions from J into $\mathbb{R}$. For any $n \in \mathbb{N}$, we denote by $AC^n(J)$ the space defined by

$$AC^n(J) := \left\{f : J \to \mathbb{R} : \frac{d^n}{dt^n} f(t) \in AC(J, \mathbb{R})\right\}.$$

Consider the space $X_b^p(T_1, T_2)$ $(b \in \mathbb{R},\ 1 \le p \le \infty)$ of those real-valued Lebesgue measurable functions f on $[T_1, T_2]$, for which $\|f\|_{X_b^p} < \infty$, where the norm is defined by

$$\|f\|_{X_b^p} = \left(\int_{T_1}^{T_2} |t^b f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}.$$

Consider the Banach space

$$\mathcal{PC}(J,\mathbb{R}) = \Big\{x : t \to \mathbb{R} : \ x \in C\left(J_k, \mathbb{R}\right); k = 0, \ldots, m, \text{ and there exist } x\left(t_k^-\right) \text{ and } x\left(t_k^+\right); k = 1, \ldots, m, \quad \text{with } \ x\left(t_k^-\right) = x\left(t_k\right)\Big\},$$

with the norm

$$\|x\|_{\mathcal{PC}} = \sup_{t\in J} |x(t)|.$$

Definition 1 (The Riemann-Liouville tempered fractional integral [19, 37, 38])**.** Suppose that the real function f is piecewise continuous on $[T_1, T_2]$ and $f \in X_b^p(T_1, T_2)$, $\lambda \geq 0$. Then, the Riemann-Liouville tempered fractional integral of order α is defined by

$$\begin{aligned} {}_{T_1}I_t^{\alpha,\lambda} f(t) &= e^{-\lambda t} \ {}_{T_1}I_t^{\alpha}\left(e^{\lambda t} f(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_{T_1}^{t} \frac{e^{-\lambda(t-\tau)} f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \end{aligned} \tag{3}$$

where ${}_{T_1}\mathcal{I}_t^{\alpha}$ denotes the Riemann-Liouville fractional integral [21], defined by

$$ {}_{T_1}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{T_1}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{4}$$

Obviously, the tempered fractional integral (3) reduces to the Riemann-Liouville fractional integral (4) if $\lambda = 0$.

Definition 2 (The Riemann-Liouville tempered fractional derivative [19, 37])**.** For $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \geq 0$. The Riemann-Liouville tempered fractional derivative is defined by

$$\begin{aligned} {}_{T_1}\mathfrak{D}_t^{\alpha,\lambda} f(t) &= e^{-\lambda t} {}_{T_1}\mathfrak{D}_t^{\alpha}\left(e^{\lambda t} f(t)\right) \\ &= \frac{e^{-\lambda t}}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{T_1}^{t} \frac{e^{\lambda\tau} f(\tau)}{(t-\tau)^{\alpha-n+1}} dt, \end{aligned}$$

where ${}_{T_1}\mathfrak{D}_t^{\alpha}\left(e^{\lambda t} f(t)\right)$ denotes the Riemann-Liouville fractional derivative [21], given by

$$\begin{aligned} {}_{T_1}\mathfrak{D}_t^{\alpha}\left(e^{\lambda t} f(t)\right) &= \frac{d^n}{dt^n}\left({}_{T_1}\mathcal{I}_t^{n-\alpha}\left(e^{\lambda t} f(t)\right)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{T_1}^{t} \frac{\left(e^{\lambda\tau} f(\tau)\right)}{(t-\tau)^{\alpha-n+1}} d\tau. \end{aligned}$$

Definition 3 (The Caputo tempered fractional derivative [19,38]). For $n-1<\alpha<n$, $n\in\mathbb{N}$, $\lambda\geq 0$. The Caputo tempered fractional derivative is defined as

$$\begin{aligned}{}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}f(t) &= e^{-\lambda t}\,{}_{T_1}^{C}\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}f(t)\right)\\ &= \frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\int_{T_1}^{t}\frac{1}{(t-\tau)^{\alpha-n+1}}\frac{d^n\left(e^{\lambda\tau}f(\tau)\right)}{d\tau^n}d\tau,\end{aligned}$$

where ${}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}\left(e^{\lambda t}f(t)\right)$ denotes the Caputo fractional derivative [21], given by

$${}_{T_1}^{C}\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}f(t)\right)=\frac{1}{\Gamma(n-\alpha)}\int_{T_1}^{t}\frac{1}{(t-\tau)^{\alpha-n+1}}\frac{d^n\left(e^{\lambda\tau}f(\tau)\right)}{d\tau^n}d\tau.$$

Lemma 1 ([19]). *For a constant C,*

$${}_{T_1}\mathfrak{D}_t^{\alpha,\lambda}C=Ce^{-\lambda t}{}_{T_1}\mathfrak{D}_t^{\alpha}e^{\lambda t},\quad {}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}C=Ce^{-\lambda t}\,{}_{T_1}^{C}\mathfrak{D}_t^{\alpha}e^{\lambda t}.$$

Obviously, ${}_{T_1}\mathfrak{D}_t^{\alpha,\lambda}(C)\neq{}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}(C)$. And, ${}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}(C)$ is no longer equal to zero, being different from ${}_{T_1}^{C}\mathfrak{D}_t^{\alpha}(C)\equiv 0$.

Lemma 2 ([19,38]). *Let $f(t)\in AC^n[T_1,T_2]$, $\lambda\geq 0$ and $n-1<\alpha<n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the following composite properties:*

$${}_{T_1}I_t^{\alpha,\lambda}\left[{}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}f(t)\right]=f(t)-\sum_{k=0}^{n-1}e^{-\lambda t}\frac{(t-T_1)^k}{k!}\left[\left.\frac{d^k\left(e^{\lambda t}f(t)\right)}{dt^k}\right|_{t=T_1}\right]$$

and

$${}_{T_1}^{C}\mathfrak{D}_t^{\alpha,\lambda}\left[{}_{a}I_t^{\alpha,\lambda}f(t)\right]=f(t),\quad \textit{for } \alpha\in(0,1).$$

Lemma 3. *Let $\Lambda\in C(J,\mathbb{R})$, $\hbar_k\in\mathbb{R}$ and $0<\alpha\leq 1$. Then, the initial value problem*

$$\begin{cases}\left({}_{0}^{C}\mathfrak{D}_t^{\alpha,\lambda}x\right)(t)=\Lambda(t), & t\in J_k;k=0,1,\dots,m,\\ \Delta x|_{t=t_k}=\hbar_k, & k=1,2,\dots,m,\\ x(0)=x_0,\end{cases}\tag{5}$$

has a unique solution defined by
(6)

$$x(t)=\begin{cases} x_0 e^{-\lambda t}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau, \quad \text{if } t\in[0,t_1], \\ e^{-\lambda t}\left[x_0\exp\left(-\lambda\displaystyle\sum_{k=1}^{k=\ell}t_k\right)+\displaystyle\sum_{k=1}^{k=k}\hbar_k\exp\left(-\lambda\sum_{\imath=k+1}^{k}t_\imath\right)\right. \\ \left.\quad+\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{k=1}^{k=k}\exp\left(-\lambda\sum_{\imath=k+1}^{k}t_\imath\right)\int_{t_{k-1}}^{t_k}e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\right] \\ \quad+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau, \quad t\in J_k,\ k=1,2,\ldots,m. \end{cases}$$

Proof. If $t\in[0,t_1]$, then applying the operator ${}_0I_t^{\alpha,\lambda}(\cdot)$ on

$$\left({}_0^C\mathfrak{D}_t^{\alpha,\lambda}x\right)(t)=\Lambda(t),$$

and by Lemma 2 and if $t\in J$, we get

$$x(t)-x(0)e^{-\lambda t}=\frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau.$$

From the initial conditions, we get

$$x(t)=x_0e^{-\lambda t}+\frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau.$$

If $t\in(t_1,t_2]$, then from Lemma 2, we have

$$\begin{aligned} x(t)&=x(t_1^+)e^{-\lambda t}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\ &=\Delta x|_{t=t_1}e^{-\lambda t}+x(t_1^-)e^{-\lambda t}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\ &=x_0e^{-\lambda(t+t_1)}+\hbar_1e^{-\lambda t}+\frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_0^{t_1}e^{-\lambda(t_1-\tau)}(t_1-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau. \end{aligned}$$

If $t\in(t_2,t_3]$, then from Lemma 2, we obtain

$$\begin{aligned} x(t)&=x(t_2^+)e^{-\lambda t}+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\ &=\Delta x|_{t=t_2}e^{-\lambda t}+x(t_2^-)e^{-\lambda t}+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \end{aligned}$$

$$
\begin{aligned}
&= \hbar_2 e^{-\lambda t} + \Bigg[x_0 e^{-\lambda(t_2+t_1)} + \hbar_1 e^{-\lambda t_2} \\
&\qquad\qquad + \frac{e^{-\lambda t_2}}{\Gamma(\alpha)} \int_0^{t_1} e^{-\lambda(t_1-\tau)}(t_1-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&\qquad\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)}(t_2-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg] e^{-\lambda t} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&= x_0 e^{-\lambda(t+t_1+t_2)} + \left[\hbar_2 e^{-\lambda t} + \hbar_1 e^{-\lambda(t+t_2)}\right] \\
&\quad + \Bigg[\frac{e^{-\lambda(t+t_2)}}{\Gamma(\alpha)} \int_0^{t_1} e^{-\lambda(t_1-\tau)}(t_1-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&\qquad + \frac{e^{-\lambda t}}{\Gamma(\alpha)} \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)}(t_2-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau.
\end{aligned}
$$

If $t \in (t_3, t_4]$, then from Lemma 2, we have

$$
\begin{aligned}
x(t) &= x(t_3^+)e^{-\lambda t} + \frac{1}{\Gamma(\alpha)} \int_{t_3}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&= \Delta x|_{t=t_3} e^{-\lambda t} + x(t_3^-)e^{-\lambda t} + \frac{1}{\Gamma(\alpha)} \int_{t_3}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&= \hbar_3 e^{-\lambda t} + \Bigg[x_0 e^{-\lambda(t_3+t_2+t_1)} + \left(\hbar_2 e^{-\lambda t_3} + \hbar_1 e^{-\lambda(t_3+t_2)}\right) \\
&\qquad\qquad + \Bigg(\frac{e^{-\lambda(t_3+t_2)}}{\Gamma(\alpha)} \int_0^{t_1} e^{-\lambda(t_1-\tau)}(t_1-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&\qquad\qquad + \frac{e^{-\lambda t_3}}{\Gamma(\alpha)} \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)}(t_2-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg) \\
&\qquad\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_3} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg] e^{-\lambda t} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_3}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau \\
&= \left[\hbar_3 e^{-\lambda t} + \hbar_2 e^{-\lambda(t+t_3)} + \hbar_1 e^{-\lambda(t+t_2+t_3)}\right] + x_0 e^{-\lambda(t+t_1+t_2+t_3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{e^{-\lambda(t+t_2+t_3)}}{\Gamma(\alpha)}\int_0^{t_1} e^{-\lambda(t_1-\tau)}(t_1-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\right.\\
&\qquad + x_0e^{-\lambda(t+t_1+t_2+t_3)} + \frac{e^{-\lambda(t+t_3)}}{\Gamma(\alpha)}\int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)}(t_2-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\\
&\qquad \left.+ x_0e^{-\lambda(t+t_1+t_2+t_3)} + \frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_{t_2}^{t_3} e^{-\lambda(t_3-\tau)}(t_3-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_3}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau.
\end{aligned}
$$

Continuing this process, we get the solution $x(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, 2, \dots, m$. Hence,

$$
\begin{aligned}
x(t) = e^{-\lambda t}&\left[x_0 \exp\left(-\lambda\sum_{k=1}^{k=k} t_k\right) + \sum_{k=1}^{k=k} \hbar_k \exp\left(-\lambda \sum_{\imath=k+1}^{k} t_\imath\right)\right.\\
&\left.+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{k=k}\exp\left(-\lambda\sum_{\imath=k+1}^{k} t_\imath\right)\int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau.
\end{aligned}
$$

By Lemma 1 and Lemma 2, if x verifies equation (6), then it satisfied the problem (5). □

Definition 4 ([7])**.** Let X be a set and $\varepsilon \geq 1$. A distance function $d : X \times X \to [0, \infty)$ is called a b-metric if the following conditions hold for all $x_1, x_2, x_3 \in X$:

(1) $d(x_1, x_2) = 0$, if and only if $x_1 = x_2$,
(2) $d(x_1, x_2) = d(x_2, x_1)$,
(3) $d(x_1, x_2) \leq \varepsilon[d(x_1, x_3) + d(x_3, x_2)]$ (b-triangular inequality).

Then, the pair (X, d, ε) is called a b-metric space with parameter ε.

Let $\overline{\Lambda}$ be the set of all increasing and continuous function $\delta : [0, \infty) \to [0, \infty)$ satisfying: $\delta(\varepsilon x) \leq \varepsilon\delta(x) \leq \varepsilon x$, for $\varepsilon > 1$ and $\delta(0) = 0$. We denote by $\Bbbk$ the family of all nondecreasing functions $\eta : (0, \infty) \to [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 5 ([7])**.** Let (X, d, ε) be a b-metric space, $\mathfrak{S} : X \to X$ is said to be a generalized ω-δ-Geraghty mapping whenever there exists $\omega : X \times X \to (0, \infty)$ such that

$$\omega(x_1, x_2)\delta\big(\varepsilon^3 d\big(\mathfrak{S}(x_1), \mathfrak{S}(x_2)\big)\big) \leq \eta\big(\delta(d(x_1, x_2))\big)\delta(d(x_1, x_2)),$$

for $x_1, x_2 \in X$, where $\eta \in \Bbbk$.

Definition 6 ([7])**.** Let X be a non empty set, $\mathfrak{S} : X \to X$ and $\omega : X \times X \to (0, \infty)$ be given mappings. The operator $\mathfrak{S}$ is orbital ω-admissible if for $x \in X$, we have

$$\omega(x, \mathfrak{S}(x)) \geq 1 \quad \Rightarrow \quad \omega\big(\mathfrak{S}(x), \mathfrak{S}^2(x)\big) \geq 1.$$

Definition 7 ([9])**.** A mapping $\Im : X \to X$ is said to be a generalized nonlinear $F-$contraction if there exist the functions $F : (0, \infty) \to \mathbb{R}$ and $\sigma : (0, \infty) \to (0, \infty)$ such that for all $x, \aleph \in X$ such that $\Im x \neq \Im \aleph$.

$$\sigma(d(x, \aleph)) + F(\varpi d(\Im x, \Im \aleph)) \leq F(A^{\epsilon d}(x, \aleph)), \tag{7}$$

where $\varpi > 1$, $\beta \in [0, 1]$ and

$$A^{\epsilon d}(x, \aleph) = \max\Big\{d(x, \aleph), d(x, \Im x), d(\aleph, \Im \aleph), \frac{\beta}{2\epsilon}\big[d(\aleph, \Im x) + d(x, \Im \aleph)\big]\Big\}.$$

Theorem 1 ([6], Corollary 3.1)**.** *Let (X, d) be a complete b-metric space and $\Im : X \to X$ be a generalized ω-δ-Geraghty mapping where1*

- (a)**:** *$\Im$ is ω-admissible;*
- (b)**:** *there exists $x_0 \in X$ where $\omega(x_0, \Im(x_0)) \geq 1$;*
- (c)**:** *if $(x_n)_{n \in N} \subset X$ with $x_n \to x$ and $\omega(x_n, x_{n+1}) \geq 1$, then $\omega(x_n, x) \geq 1$,*

then $\Im$ has a fixed point. Moreover, if

- (d)**:** *for all fixed points $x, \widehat{x}$ of $\Im$, either*

$$\omega(x, \widehat{x}) \geq 1 \quad or \quad \omega(\widehat{x}, x) \geq 1,$$

then $\Im$ has a unique fixed point.

Theorem 2 ([9], Theorem 12)**.** *Let (X, d, ϵ) be a complete b-metric space. A generalized nonlinear F-contraction $\Im$ has a fixed point if the following statements are true:*

- (1)**:** *$\Im$ is increasing, that is, if $a < b$, then $\Im(a) < \Im(b)$, for all $a, b \in (0, \infty)$;*
- (2)**:** $\beta < 1$;
- (3)**:** $\frac{\varepsilon}{\alpha} < 1$;
- (4)**:** $\liminf_{x \to t^+} \sigma(x) > 0$, *for any $t \geq 0$.*

3. Existence of solutions for the first problem

Let $(\mathcal{PC}(J, \mathbb{R}), d, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $d : \mathcal{PC}(J, \mathbb{R}) \times \mathcal{PC}(J, \mathbb{R}) \to [0, \infty)$, is given by:

$$d(x, \aleph) = \|(x - \aleph)^2\|_{\mathcal{PC}} := \sup_{t \in J} |x(t) - \aleph(t)|^2.$$

In this section, we establish some existence results for problem (1).

Definition 8. By a solution of problem (1), we mean a function $x \in \mathcal{PC}(J,\mathbb{R})$ given by
(8)

$$x(t) = e^{-\lambda t}\Bigg[x_0 \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) + \sum_{0<t_\imath<t} \hbar_k(x(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big)$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau,$$

where $\Lambda \in \mathcal{PC}(J,\mathbb{R})$ such that $\Lambda(t) = f(t, x(t), \Lambda(t))$.

The hypotheses:

(H_1)**:** There exist continuous functions $\gamma_1 : J \to (0,\infty)$ and $\gamma_2 : J \to (0,1)$ and $\vartheta > 0$ such that for each $x, \aleph, x_1, \aleph_1 \in \mathcal{PC}(J,\mathbb{R})$ and $t \in J$, we have

$$|f(t,x(t),\aleph(t)) - f(t,x_1(t),\aleph_1(t))|$$
$$\leq \gamma_1(t)|x(t)-x_1(t)| + \gamma_2(t)|\aleph(t)-\aleph_1(t)|$$

and

$$|\hbar_k(x(t)) - \hbar_k(\aleph(t))| \leq \vartheta|x(t)-\aleph(t)|,$$

with

$$\Bigg\|m\vartheta + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)} \frac{{\gamma_1}^*}{1-\gamma_2*}d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)} \frac{{\gamma_1}^*}{1-\gamma_2*}d\tau\Bigg\|_{\mathcal{PC}}^2 \leq \delta(\|(x-\aleph)^2\|_{\mathcal{PC}})$$

and

$${\gamma_1}^* = \|\gamma_1\|_\infty, \quad {\gamma_2}^* = \|\gamma_2\|_\infty.$$

(H_2)**:** There exist $\delta \in \overline{\Lambda}$ and $\overline{\lambda}_0 \in \mathcal{PC}(J,\mathbb{R})$ and a function $\wp : \mathcal{PC}(J,\mathbb{R}) \times \mathcal{PC}(J,\mathbb{R}) \to \mathbb{R}$, such that

$$\wp\Bigg(\overline{\lambda}_0(t), e^{-\lambda t}\Big[\overline{\lambda}_0 \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) + \sum_{0<t_k<t} \hbar_k(\overline{\lambda}_0(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big)$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Big]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Bigg) \geq 0,$$

where $\Lambda \in \mathcal{PC}(J,\mathbb{R})$ such that $\Lambda(t) = f(t, \overline{\lambda}_0(t), \Lambda(t))$.

(H_3): For each $t \in J$ and $x, \aleph \in \mathcal{P}C(J, \mathbb{R})$, we have

$$\wp(x(t), \aleph(t)) \geq 0$$

implies

$$\wp\Big(e^{-\lambda t}\Big[x_0 \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) + \sum_{0<t_k<t} \hbar_k(x(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big)$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Big]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau,$$

$$e^{-\lambda t}\Big[\aleph_0 \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) + \sum_{0<t_k<t} \hbar_k(\aleph(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big)$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\widehat{\Lambda}(\tau)d\tau\Big]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\widehat{\Lambda}(\tau)d\tau\Big) \geq 0,$$

where $\Lambda, \widehat{\Lambda} \in \mathcal{P}C(J, \mathbb{R})$ such that

$$\Lambda(t) = f(t, x(t), \Lambda(t))$$

and

$$\widehat{\Lambda}(t) = f(t, \aleph(t), \widehat{\Lambda}(t)).$$

(H_4): If $(x_n)_{n\in N} \subset \mathcal{P}C(J, \mathbb{R})$ with $x_n \to x$ and $\wp(x_n, x_{n+1}) \geq 1$, then

$$\wp(x_n, x) \geq 1.$$

(H_5): For all fixed solutions $x, \widehat{x}$ of problem (1), either

$$\wp(x(t), \widehat{x}(t)) \geq 0$$

or

$$\wp(\widehat{x}(t), x(t)) \geq 0.$$

Theorem 3. *Assume that the hypotheses* (H_1)-(H_4) *hold. Then, the problem* (1) *has a least one solution defined on* J. *Moreover, if* (H_5) *holds, then we get a unique solution.*

Proof. Consider the operator $\Upsilon : \mathcal{PC}(J,\mathbb{R}) \to \mathcal{PC}(J,\mathbb{R})$ defined by:

$$
\begin{aligned}
(\Upsilon x)(t) = e^{-\lambda t}\Big[&x_0 \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) \\
&+ \sum_{0<t_k<t} \hbar_k(x(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&\times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Big] \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau,
\end{aligned}
\tag{9}
$$

where $\Lambda \in \mathcal{PC}(J,\mathbb{R})$ such that $\Lambda(t) = f(t, x(t), \Lambda(t))$.

The function $\omega : \mathcal{PC}(J,\mathbb{R}) \times \mathcal{PC}(J,\mathbb{R}) \to [0,\infty)$ is given by:

$$
\begin{cases}
\omega(x,\widehat{x}) = 1, & \text{if } \wp(x(t),\widehat{x}(t)) \geq 0,\ t \in J, \\
\omega(x,\widehat{x}) = 0, & \text{else.}
\end{cases}
$$

First, we prove that Υ is a generalized ω-δ-Geraghty operator: For any $x, \widehat{x} \in \mathcal{PC}(J,\mathbb{R})$ and each $t \in J$, we have

$$
\begin{aligned}
&|(\Upsilon x)(t) - (\Upsilon \widehat{x})(t)| \\
&\leq e^{-\lambda t}\Big[\sum_{0<t_k<t} |\hbar_k(x(t_k^-)) - \hbar_k(\widehat{x}(t_k^-))| \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&\quad \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}|\Lambda(\tau) - \widehat{\Lambda}(\tau)|d\tau\Big] \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}|\Lambda(\tau) - \widehat{\Lambda}(\tau)|d\tau,
\end{aligned}
$$

where $\Lambda, \widehat{\Lambda} \in \mathcal{PC}(J,\mathbb{R})$ such that

$$
\Lambda(t) = f\big(t, x(t), \Lambda(t)\big) \quad \text{and} \quad \widehat{\Lambda}(t) = f\big(t, \widehat{x}(t), \widehat{\Lambda}(t)\big).
$$

From (H_1) we have

$$
\|\Lambda - \widehat{\Lambda}\|_\infty \leq \frac{{\gamma_1}^*}{1-\gamma_2*}\|(x-\widehat{x})^2\|_{\mathcal{PC}}^{\frac{1}{2}},
$$

where $\gamma_1* = \sup_{t\in J}|\gamma_1(t)|$ and $\gamma_2* = \sup_{t\in J}|\gamma_2(t)|$.

Next, we have

$$
\begin{aligned}
&|(\Upsilon x)(t)-(\Upsilon \widehat{x})(t)| \\
&\leq m\vartheta \|(x-\widehat{x})^2\|_{\mathcal{P}C}^{\frac{1}{2}} \\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\frac{\gamma_1{}^*}{1-\gamma_2*}\|(x-\widehat{x})^2\|_{\mathcal{P}C}^{\frac{1}{2}}d\tau \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\frac{\gamma_1{}^*}{1-\gamma_2*}\|(x-\widehat{x})^2\|_{\mathcal{P}C}^{\frac{1}{2}}d\tau.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\omega(x,x_2')|(\Upsilon x_2)(t)-(\Upsilon x_2')(t)|^2 \\
&\leq \Bigg\| m\vartheta+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\frac{\gamma_1{}^*}{1-\gamma_2*}d\tau \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\frac{\gamma_1{}^*}{1-\gamma_2*}d\tau\Bigg\|_{\mathcal{P}C}^2 \times \left\|(x-\widehat{x})^2\right\|_{\mathcal{P}C}\omega(x,\widehat{x}) \\
&\leq \|(x-\widehat{x})^2\|_{\mathcal{P}C}\delta(\|(x-\widehat{x})^2\|_{\mathcal{P}C}).
\end{aligned}
$$

Hence,

$$
\omega(x,\widehat{x})\delta\big(2^3 d(\Upsilon(x),\Upsilon(\widehat{x}))\big) \leq \eta\big(\delta(d(x,\widehat{x}))\big)\delta(d(x,\widehat{x})),
$$

where $\eta\in\mathbb{k}$, $\delta\in\overline{\Lambda}$, with $\eta(t)=\frac{1}{8}t$, and $\delta(t)=t$. So, Υ is generalized ω-δ-Geraghty operator.

Let $x,\widehat{x}\in\mathcal{P}C(J,\mathbb{R})$ such that

$$
\omega(x,\widehat{x})\geq 1.
$$

Thus, for each $t\in J$, we have

$$
\wp(x(t),\widehat{x}(t))\geq 0.
$$

This implies from (H_3) that

$$
\wp(\Upsilon x(t),\Upsilon\widehat{x}(t))\geq 0,
$$

which gives

$$
\omega(\Upsilon(x),\Upsilon(\widehat{x}))\geq 1.
$$

Hence, Υ is a ω-admissible.

Now, by (H_2), there exist $\overline{\lambda}_0\in C(J,\mathbb{R})$ such that

$$
\omega(\overline{\lambda}_0,\Im(\overline{\lambda}_0))\geq 1.
$$

Thus, by (H_4), if $\left(\overline{\lambda}_n\right)_{n\in N}\subset X$ with $\overline{\lambda}_n\to\overline{\lambda}$ and $\omega(\overline{\lambda}_n,\overline{\lambda}_{n+1})\geq 1$, then

$$
\omega(\overline{\lambda}_n,\overline{\lambda})\geq 1.
$$

From an application of Theorem 1(a)-(c), we conclude that Υ has a fixed point x which is a solution of problem (1).

Moreover, (H_5) implies that if x and $\widehat{x}$ are fixed points of Υ, then either

$$\wp(x,\widehat{x}) \geq 0 \quad \text{or} \quad \wp(\widehat{x},x) \geq 0.$$

This implies that either

$$\omega(x,\widehat{x}) \geq 1 \quad \text{or} \quad \omega(\widehat{x},x) \geq 1.$$

Hence, problem (1) has a unique solution by Theorem 1(d). □

Now, we prove the existence and uniqueness results by using the F-contraction fixed point theorem.

Theorem 4. *Assume that there exist constants $\widehat{\omega}, \widetilde{\omega} > 0$, where $\overline{\omega} = \widehat{\omega}(1-\widetilde{\omega}) > \sqrt{2}$ such that for each $x, \aleph, x_1, \aleph_1 \in \mathcal{PC}(J,\mathbb{R})$ and $t \in J$, we have*

$$\begin{aligned} \big|f(t,x(t),\aleph(t)) - f(t,x_1(t),\aleph_1(t))\big| &\leq \widetilde{\omega}\big|\aleph(t)-\aleph_1(t)\big| \\ + \frac{\Gamma(\alpha+1)\big|x(t)-x_1(t)\big|}{2\widehat{\omega}(\Gamma(\alpha+1)m\vartheta+(m+1)T^{\alpha})\big[1+\sup_{t\in J}|x(t)|+\sup_{t\in J}|\aleph(t)|\big]} \end{aligned} \tag{10}$$

and

$$\big|\hbar_k(x(t)) - \hbar_k(\aleph(t))\big| \leq \vartheta\big|x(t)-\aleph(t)\big|.$$

Then, the problem (1) *has a unique solution.*

Proof. Let $\Upsilon : \mathcal{PC}(J,\mathbb{R}) \to \mathcal{PC}(J,\mathbb{R})$ defined as in (9), for any x, $\widehat{x} \in \mathcal{PC}(J,\mathbb{R})$. For each $t \in J$, we have

$$\begin{aligned} &\big|(\Upsilon x)(t) - (\Upsilon\widehat{x})(t)\big|^2 \\ &\leq \Bigg\{e^{-\lambda t}\Big[\sum_{0<t_k<t}\big|\hbar_k(x_{t_k^-}) - \hbar_k(\widehat{x}_{t_k^-})\big|\exp\Big(-\lambda\sum_{t_{k+1}<t_\imath<t}t_\imath\Big) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\exp\Big(-\lambda\sum_{t_{k+1}<t_\imath<t}t_\imath\Big) \\ &\quad \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}|\Lambda(\tau)-\widehat{\Lambda}(\tau)|d\tau\Big] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}|\Lambda(\tau)-\widehat{\Lambda}(\tau)|d\tau\Bigg\}^2, \end{aligned}$$

where Λ, $\widehat{\Lambda} \in \mathcal{PC}(J,\mathbb{R})$, and

$$\Lambda(t) = f(t,x(t),\Lambda(t)) \quad \text{and} \quad \widehat{\Lambda}(t) = f(t,x(t),\widehat{\Lambda}(t)).$$

For each $t \in J$, we have

$$\begin{aligned} \|\Lambda - \widehat{\Lambda}\|_\infty &\leq |x(t)-\widehat{x}(t)| \\ \times &\frac{\Gamma(\alpha+1)}{2(1-\widetilde{\omega})\widehat{\omega}(\Gamma(\alpha+1)m\vartheta+(m+1)\big[1+\sup_{t\in J}|\varkappa_2(t)|+\sup_{t\in J}|\varkappa_2'(t)|\big]}. \end{aligned}$$

Then, we get

$$
\begin{aligned}
&|(\Upsilon x)(t)-(\Upsilon\widehat{x})(t)|^2\\
&\le \Big\{\frac{1}{\varpi\big[2+2\sup_{t\in J}|x(t)|+\sup_{t\in J}|\widehat{x}(t)|\big]}|x(t)-\widehat{x}(t)|\Big\}^2\\
&\le \frac{1}{\varpi^2\big[2+2\sup_{t\in J}|x(t)|+\sup_{t\in J}|\widehat{x}(t)|\big]^2}\Big\{\sqrt{|x(t)-\widehat{x}(t)|^2}\Big\}^2\\
&\le \frac{1}{\varpi^2\big[2+2\sup_{t\in J}|x(t)|+\sup_{t\in J}|\widehat{x}(t)|\big]^2}\Big\{\sqrt{\sup_{t\in J}|x(t)-\widehat{x}(t)|^2}\Big\}^2\\
&\le \frac{1}{\varpi^2\big[2+2\sup_{t\in J}|x(t)|-\widehat{x}(t)\big]^2}\Big\{\sqrt{\sup_{t\in J}|x(t)-\widehat{x}(t)|^2}\Big\}^2.
\end{aligned}
$$

Consequently, we get

$$\varpi^2 d(\Upsilon x,\Upsilon\widehat{x})\le \frac{d(x,\widehat{x})}{2+d(x,\widehat{x})}.$$

Now, applying natural logarithm on the previous inequality, we obtain

$$\ln(2+d(x,\widehat{x}))+\ln(\varpi^2 d(\Upsilon x,\Upsilon\widehat{x})) \;\le\; \ln(d(x,\widehat{x})) \;\le\; \ln(A^{\epsilon d}(x,\widehat{x})),$$

where $\beta<\frac{1}{2}$ and

$$A^{\epsilon d}(x,\widehat{x})=\max\Big\{d(x,\widehat{x}),d(x,\Upsilon x),d(\widehat{x},\Upsilon\widehat{x}),\frac{\beta}{2\epsilon}\big[d(\widehat{x},\Upsilon x)+d(x,\Upsilon\widehat{x})\big]\Big\}.$$

If we choose $F(t)=\ln(t)$ and $\sigma(t)=\ln(2+t)$ we see that all the conditions of Theorem 2 are satisfied, so that Υ has a unique fixed point which is the solution of problem. □

4. Existence of solutions for the second problem

In this section, we establish some existence results for problem (2).

Definition 9. By a solution of problem (2), we mean a continuous function $x\in\mathcal{PC}(J,\mathbb{R})$ given by

$$
\begin{aligned}
x(t)=e^{-\lambda t}\Big[&(x_0-\overline{\delta}(x))\exp\Big(-\lambda\sum_{0<t_k<t}t_k\Big)\\
&+\sum_{0<t_\imath<t}\hbar_k(x_{t_k^-})\exp\Big(-\lambda\sum_{t_{k+1}<t_\imath<t}t_\imath\Big)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\exp\Big(-\lambda\sum_{t_{k+1}<t_\imath<t}t_\imath\Big)\\
&\times\int_{t_{k-1}}^{t_k}e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Big]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau,
\end{aligned}
\tag{11}
$$

where $\Lambda\in\mathcal{PC}(J,\mathbb{R})$ such that $\Lambda(t)=f(t,x(t),\Lambda(t))$.

Let us introduce the following hypotheses:

(H_6) There exists a constant $\varsigma > 0$ such that

$$|\overline{\delta}(x) - \overline{\delta}(\aleph)| \leq \varsigma \|x - \aleph\|_\infty,$$

for each $x, \aleph \in \mathcal{PC}(J)$.

(H_7) There exist $\overline{M} : J \to (0, \infty)$ and $\overline{N} : J \to (0, 1)$ and $\overline{\vartheta} > 0$ such that for each $x, \aleph, x_1, \aleph_1 \in \mathbb{R}$ and $t \in J$, we have

$$\left| f(t, x, \aleph) - f(t, x_1, \aleph_1) \right| \leq \overline{M}(t)|x - x_1| + \overline{N}(t)|\aleph - \aleph_1|$$

and

$$|\hbar_k(x) - \hbar_k(\aleph)| \leq \overline{\vartheta}|x - \aleph|,$$

with

$$\Bigg\| m\overline{\vartheta} + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k - \tau)^{(\alpha-1)} \frac{\overline{M}^*}{1 - \overline{N}^*} d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t - \tau)^{(\alpha-1)} \frac{\overline{M}^*}{1 - \overline{N}^*} d\tau \Bigg\|_{\mathcal{PC}}^2$$

$$\leq \delta(\|(x - \aleph)^2\|_{\mathcal{PC}}).$$

(H_8) There exist $\delta \in \overline{\Lambda}$ and $\overline{\lambda}_0 \in \mathcal{PC}(J, \mathbb{R})$ and a function $\wp : \mathcal{PC}(J, \mathbb{R}) \times \mathcal{PC}(J, \mathbb{R}) \to \mathbb{R}$, such that

$$\wp\Bigg(\overline{\lambda}_0(t), e^{-\lambda t}\Bigg[(\overline{\lambda}_{0_0} - \overline{\delta}(\overline{\lambda}_0)) \exp\Big(-\lambda \sum_{0<t_k<t} t_k \Big)$$

$$+ \sum_{0<t_k<t} \hbar_k(\overline{\lambda}_0(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath \Big)$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath \Big)$$

$$\cdot \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k - \tau)^{(\alpha-1)} \Lambda(\tau) d\tau \Bigg]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t - \tau)^{(\alpha-1)} \Lambda(\tau) d\tau \Bigg) \geq 0,$$

where $\Lambda \in \mathcal{PC}(J, \mathbb{R})$ such that $\Lambda(t) = f(t, \overline{\lambda}_0(t), \Lambda(t))$.

(H_9) For each $t \in J$, and $x, \aleph \in \mathcal{PC}(J, \mathbb{R})$, we have

$$\wp(x(t), \aleph(t)) \geq 0$$

implies

$$\begin{aligned}
\wp\Bigg(& e^{-\lambda t}\Big[(x_0 - \overline{\delta}(x)) \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) \\
& \quad + \sum_{0<t_k<t} \hbar_k(x(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
& \quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
& \qquad \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)} (t_k-\tau)^{(\alpha-1)} \Lambda(\tau) d\tau\Big] \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)} (t-\tau)^{(\alpha-1)} \Lambda(\tau) d\tau, \\
& e^{-\lambda t}\Big[(\aleph_0 - \overline{\delta}(\aleph)) \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) \\
& \quad + \sum_{0<t_k<t} \hbar_k(\aleph(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
& \quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
& \qquad \times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)} (t_k-\tau)^{(\alpha-1)} \widehat{\Lambda}(\tau) d\tau\Big] \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)} (t-\tau)^{(\alpha-1)} \widehat{\Lambda}(\tau) d\tau \Bigg) \geq 0,
\end{aligned}$$

where $\Lambda, \widehat{\Lambda} \in \mathcal{PC}(J, \mathbb{R})$ such that

$$\Lambda(t) = f(t, x(t), \Lambda(t))$$

and

$$\widehat{\Lambda}(t) = f(t, \aleph(t), \widehat{\Lambda}(t)).$$

(H_{10}) For all fixed solutions $x, \widehat{x}$ of problem (2), either

$$\wp(x(t), \widehat{x}(t)) \geq 0$$

or

$$\wp(\widehat{x}(t), x(t)) \geq 0.$$

Theorem 5. *Assume that the hypotheses* (H_4) *and* (H_6)*-*(H_9) *hold. Then the problem* (2) *has a least one solution defined on* J*. Moreover, if* (H_{10}) *holds, then we get a unique solution.*

Proof. Consider the operator $\Upsilon' : \mathcal{PC}(J, \mathbb{R}) \to \mathcal{PC}(J, \mathbb{R})$ defined by:

$$
\begin{aligned}
(\Upsilon' x)(t) = e^{-\lambda t}\Big[&(x_0 - \overline{\delta}(x)) \exp\Big(-\lambda \sum_{0<t_k<t} t_k\Big) \\
&+ \sum_{0<t_k<t} \hbar_k(x(t_k^-)) \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \exp\Big(-\lambda \sum_{t_{k+1}<t_\imath<t} t_\imath\Big) \\
&\times \int_{t_{k-1}}^{t_k} e^{-\lambda(t_k-\tau)}(t_k-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau\Big] \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} e^{-\lambda(t-\tau)}(t-\tau)^{(\alpha-1)}\Lambda(\tau)d\tau,
\end{aligned}
\tag{12}
$$

where $\Lambda \in \mathcal{PC}(J, \mathbb{R})$ such that $\Lambda(t) = f(t, x(t), \Lambda(t))$.

Clearly, the fixed points of the operator Υ' are solution of the problem (2). By repeating the same process of Theorem 3, we can easily show all the conditions of Theorem 5 are satisfied by Υ'. Since the proof is standard, we omit it here. □

Now, we prove an existence and uniqueness result by using the F-contraction fixed point theorem.

Theorem 6. *Assume that the hypothesis* (H_6) *holds and there exist constants* $\varpi'', \varpi' > 0$*, where* $\overline{\varpi} = \varpi''(1-\varpi') > \sqrt{2}$ *such that for each* $x, \aleph, x_1, \aleph_1 \in \mathcal{PC}(J, \mathbb{R})$ *and* $t \in J$*, we have*

$$
\begin{aligned}
|f(t, x(t), \aleph(t)) - f(t, x_1(t), \aleph_1(t))| \;&\le\; \varpi'|\aleph(t) - \aleph_1(t)| + \\
&\frac{\Gamma(\alpha+1)|x(t) - x_1(t)|}{2\varpi''(\Gamma(\alpha+1)(\varsigma + m\overline{\vartheta}) + (m+1)T^{\alpha})\left[1 + \sup_{t\in J}|x(t)| + \sup_{t\in J}|\aleph(t)|\right]}
\end{aligned}
\tag{13}
$$

and

$$
|\hbar_k(x(t)) - \hbar_k(\aleph(t))| \;\le\; \overline{\vartheta}|x(t) - \aleph(t)|.
$$

Then, the problem (2) *has a unique solution.*

Proof. Consider the operator $\Upsilon' : \mathcal{PC}(J, \mathbb{R}) \to \mathcal{PC}(J, \mathbb{R})$ defined an in (12).

Clearly, the fixed points of the operator Υ' are solution of the problem (2). By repeating the same process of Theorem 4, we can easily show all the conditions of Theorem 6 are satisfied by Υ'. Since the proof is standard, we omit it here. □

5. Some examples

Example 1. Consider the following problem which is an example of our problem (1)):

$$\begin{cases} \left({}_0^C\mathfrak{D}_t^{\frac{1}{2},\lambda}x\right)(t) = \dfrac{\sin(t)}{90(1+|x(t)|)} \\ \qquad\qquad + \dfrac{e^{-t}}{180\left(1+\left|\left({}_0^C\mathfrak{D}_t^{\frac{1}{2},\lambda}x\right)(t)\right|\right)}, & t \in J_0 \cup J_1, \\ \Delta x|_{t=t_k} = \dfrac{x(t_k^-)}{30+x(t_k^-)}, & k=1, \\ x(0) = 0, \end{cases} \tag{14}$$

where $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$ and $t_1 = 1$.

Set

$$f(t, x, \aleph) \;=\; \frac{\sin(t)}{90(1+|x|)} + \frac{e^{-t}}{180(1+|\aleph|)},$$

where $t \in [0,1],\ x, \aleph \in \mathbb{R}$.

Let $(\mathcal{PC}([0,1]), d, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $d : \mathcal{PC}([0,1]) \times \mathcal{PC}([0,1]) \to (0, \infty)$, is given by:

$$d(x, \aleph) \;=\; \|(x-\aleph)^2\|_{\mathcal{PC}} \;:=\; \sup_{t\in[0,1]} |x(t) - \aleph(t)|^2.$$

For any $x, \bar{x}, \aleph, \bar{\aleph} \in \mathcal{PC}([0,1])$ and $t \in [0,1]$, if $|x(t)| \le |\aleph(t)|$, then

$$|f(t, x(t), \bar{x}(t)) - f(t, \aleph(t), \bar{\aleph}(t))| \;\le\; \frac{\|x-\aleph\|_{\mathcal{PC}}}{90} + \frac{\|\bar{x}-\bar{\aleph}\|_{\mathcal{PC}}}{180}$$

and

$$|\hbar_k(x(t)) - \hbar_k(\aleph(t))| \;\le\; \frac{1}{30}\|x-\aleph\|_{\mathcal{PC}}.$$

Thus, hypothesis (H_1) is satisfied with:

$$\gamma_1(t) = \frac{\sin(t)}{90}, \qquad \gamma_2(t) = \frac{e^{-t}}{180}, \qquad \vartheta = \frac{1}{30}.$$

Define the functions $\eta(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : \mathcal{PC}([0,1]) \times \mathcal{PC}([0,1]) \to \mathbb{R}_+^*$ with

$$\begin{cases} \alpha(x, \aleph) = 1, & \text{if } \ d(x(t), \aleph(t)) \ge 0,\ t \in J, \\ \alpha(x, \aleph) = 0, & \text{else,} \end{cases}$$

and $d : \mathcal{PC}([0,1]) \times \mathcal{PC}([0,1]) \to \mathbb{R}$ with $d(x, \aleph) = \|x - \aleph\|_{\mathcal{PC}}$.

Hypothesis (H_2) is satisfied with $\overline{\mu}_0(t) = x_0$. Also, (H_3) holds from the definition of the function d.

Simple computations show that all conditions of Theorem 3 are satisfied. Hence, we get the existence and the uniqueness of solutions for problem (14).

Example 2. Consider the following problem:

$$
(15) \quad \begin{cases} \left({}_0^C\mathfrak{D}_t^{\frac{1}{2},\lambda} x \right)(t) = \dfrac{\Gamma(\frac{3}{2})}{4\left(\Gamma(\frac{3}{2})\frac{6}{5}+2\right)(1+\|x\|_{\mathcal{PC}})}|x(t)| \\ \qquad + \dfrac{1}{20\left(1+\left|\left({}_0^C\mathfrak{D}_t^{\frac{1}{2},\lambda} x\right)(t)\right|\right)}, & \text{if } t \in J_0 \cup J_1, \\ \Delta x|_{t=t_k} = \frac{x(t_k^-)}{5+x(t_k^-)}, & k=1, \\ x(0) + x = 0, \end{cases}
$$

where $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$ and $t_1 = 1$.

Set

$$
f(t, x(t), \aleph(t)) = \frac{\Gamma(\frac{3}{2})}{4\left(\Gamma(\frac{3}{2})\frac{6}{5}+2\right)(1+\|x\|_{\mathcal{PC}})}|x(t)| + \frac{1}{20(1+|\aleph(t)|)},
$$

where $t \in [0,1],\ x, \aleph \in \mathcal{PC}([0,1])$.

Let $(\mathcal{PC}([0,1]), d, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $d : \mathcal{PC}([0,1]) \times \mathcal{PC}([0,1]) \to (0, \infty)$, is given by:

$$
d(x, \aleph) = \|(x-\aleph)^2\|_{\mathcal{PC}} := \sup_{t\in[0,1]} |x(t) - \aleph(t)|^2.
$$

For any $x, \bar{x}, \aleph, \bar{\aleph} \in \mathcal{PC}([0,1])$ and $t \in [0,1]$. If $|x(t)| \le |\aleph(t)|$, then

$$
\begin{aligned}
|f(t, x(t), \bar{x}(t)) - f(t, \aleph(t), \bar{\aleph}(t))| &\le \frac{1}{20}|\bar{x}(t) - \bar{\aleph}(t)| \\
&+ \frac{\Gamma(\frac{3}{2})}{4\left(\Gamma(\frac{3}{2})\frac{6}{5}+2\right)(1+\|x\|_{\mathcal{PC}}+\|\aleph\|_{\mathcal{PC}})}|x(t) - \aleph(t)|.
\end{aligned}
$$

Then, hypothesis (13) is satisfied with

$$
\widehat{\omega} = 2, \quad \widetilde{\omega} = \frac{1}{20}, \quad \overline{\omega} = \frac{19}{10} > \sqrt{2},
$$

and hypothesis (H_6) is satisfied with

$$
\varsigma = 1.
$$

Since all requirements of Theorem 6 are verified. Then, we conclude the existence the uniqueness of solutions and for problem (15).

6. Conclusion

In the course of this research, we established both the existence and uniqueness of solutions for a class of problems involving the nonlinear implicit Caputo tempered fractional differential equations, coupled with initial nonlocal conditions. Our approach to establishing existence and uniqueness hinged on the utilization of $\omega - \delta$-Geraghty type contraction, F-contraction, and fixed point theory. In order to exemplify the applicability of our key findings and to demonstrate the feasibility of satisfying our theorem's prerequisites, we offered a range of examples. Our results in the provided context are novel and add significantly to the literature on this emerging topic of research. Due to the small amount of publications on tempered fractional calculus, we believe there are several possible study paths such as coupled systems, problems with infinite delays, and many more.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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